

About radial Toeplitz operators on Segal-Bargmann and l^2 spaces.

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We discuss Toeplitz operators on the Segal-Bargmann space as functional realizations of anti-Wick operators on the Fock space. In the special case of radial symbols we exploit the isometric mapping between the Segal-Bargmann space and l^2 complex sequences in order to establish conditions such that an equivalence between Toeplitz operators and diagonal operators on l^2 holds. We also analyze the inverse problem of mapping diagonal operators on l^2 into Toeplitz form. The composition problem of Toeplitz operators with radial symbols is reviewed as an application. Our notation and basic examples make contact with Quantum Mechanics literature.

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I. INTRODUCTION

Toeplitz operators were introduced in physics by Berezin [4] [5] [6] in the context of quantization procedures, *i.e.* in the association of a classical function in phase space and a quantum observable. In this sense, the classical function is called a symbol for the operator. Indeed, there are different ways to quantize, all of them consistent with the probabilistic interpretation of quantum mechanics. The key difference between them arises from non-commutativity of quantum observables, in contrast with commutativity of their classical counterparts. To be more specific, in a Fock space there exist annihilation and creation operators \hat{a} , \hat{a}^* which do not commute; the symbolic calculus associated to the product order $\hat{a}^*\hat{a}$ leads to Wick symbols, while that associated to the product order $\hat{a}\hat{a}^*$ leads to anti-Wick symbols. In this context, Toeplitz operators arise as the Segal-Bargmann space realizations of Fock space operators having an associated anti-Wick symbol (formal definitions are given below).

More generally, given a symbol $\varphi(z, \bar{z})$, the corresponding Toeplitz operator is defined as the Bargmann projection of the pointwise multiplication of a function in the Segal-Bargmann space with the symbol. In this sense, the operator has a natural domain not covering the Segal-Bargmann space, but restricted to functions such that the projection is well defined. This restriction poses a problem on various properties of Toeplitz operators, such as composition.

Toeplitz operators are also defined in the Bergman space of analytical functions on the unit disk, where most of the theoretical results have been developed (see for instance [8]). They have also been object of study in different disciplines. In partial differential equations, these operators and their adjoints play an important role in extending known results in the space of entire functions to the context of Segal-Bargmann spaces (see for instance [11], [10], [20], [22]). They have also been extensively studied as an efficient mathematical tool in signal analysis ([14], [15], [17]).

It is known that the composition of Toeplitz operators is in general not closed, in the sense that it may be well defined but the resulting operator is not a Toeplitz one. The problem of how to define the class of the symbols where the composition of the corresponding Toeplitz operators is closed is still open. Some authors have addressed this issue: [13] presents some classes of operators where the problem is solved, while recent extensions are presented in [2]. In contrast, in [1], [14], [15] and [23] the composition of certain Toeplitz operators has been shown to be expressed as a Toeplitz operator plus a remainder term.

This article is intended to provide a unified discussion of operators with radial anti-Wick symbols. Following insights from Quantum Mechanics, we start from the abstract Fock space formula-

tion and discuss concrete functional realizations both in Segal-Bargmann and l^2 spaces. We then apply the results to the related composition problem.

The present work is organized as follows. Section II and the Appendix contain preliminary material about Fock space and its functional realizations, in particular the isometry between the Segal-Bargmann and l^2 spaces. In Section III precise definitions of anti-Wick and Toeplitz operators are given, stressing the class of symbols and operator domains considered by different authors. Section IV addresses the case of Toeplitz operators with radial symbols. Under the isometry between the Segal-Bargmann space and l^2 sequences, some of them are unitarily equivalent to diagonal operators on l^2 . While this drastic simplification allows for a very simple analysis of their properties [18], one should notice that not any Toeplitz operator with radial symbol can be treated in this way. We provide sufficient and necessary conditions for this equivalence to hold. Section V aims to investigate some natural questions related to the the inverse problem of existence and uniqueness of an anti-Wick symbol and Toeplitz operator for a given diagonal operator on l^2 . While the general problem is hard analyze, we present a family of such operators for which the symbols can be explicitly constructed and study, on this set, sufficient and necessary conditions for unitarily equivalence between diagonal operators on l^2 and Toeplitz operators. The studied family is large enough to illustrate accomplishment or not of this conditions. Section VI discusses the composition problem, addressing to the large gap between classes of Toeplitz operators for which the composition is known to be closed and counter examples where the composition is a well defined but not a Toeplitz operator [13]. We resort to our results in Section V to present novel results on composition, and relate them to known positive results and counter-examples.

II. FUNCTIONAL REALIZATIONS OF THE FOCK SPACE

An abstract Fock space \mathcal{F} is a Hilbert space of vectors, denoted by $|\psi\rangle$ in Dirac's notation, in which there exist an operator \hat{a} and its adjoint \hat{a}^* , called annihilation and creation operators, satisfying the canonical commutation rules $[\hat{a}, \hat{a}^*] = I$, where I is the identity operator on \mathcal{F} . There also exists a vector $|0\rangle$ (called *vacuum vector*) annihilated by \hat{a} , such that the system

$$\left\{ |n\rangle = \frac{(\hat{a}^*)^n |0\rangle}{\sqrt{n!}} \right\}_{n \in \mathbb{N}} \quad (1)$$

is complete and orthonormal in \mathcal{F} (the canonical Fock space basis, known in physics as the *occupation number* basis). Then \mathcal{F} is the space of linear combinations

$$|\psi\rangle = \sum_{n \in \mathbb{N}} \psi_n |n\rangle \quad (2)$$

with complex coefficients and finite norm. The inner product is noted as $\langle\phi|\psi\rangle$, being antilinear in the left vector and linear in the right one.

Functional realizations of the Fock space are obtained by projecting vectors $|\psi\rangle$ on complete sets labeled by either a discrete or continuous variable. Classical examples of Fock space realizations are:

l^2 : projection on the orthonormal basis (1) gives simply $\langle n|\psi\rangle = \psi_n$. The finite norm condition on $|\psi\rangle$ is equivalent to $\sum_n |\psi_n|^2 < \infty$. Then the vector $|\psi\rangle$ in \mathcal{F} is realized by the sequence $\{\psi_n\}$ in l^2 , the linear space of square summable complex sequences. The inner product in l^2 is realized by

$$(\{\psi'_n\}, \{\psi_n\}) := \sum_{n \in \mathbb{N}} \overline{\psi'_n} \psi_n = \langle \psi' | \psi \rangle.$$

$L^2(\mathbb{R}, d\mathbf{x})$: consider the set $\{|x\rangle\}$ containing the generalized eigenvectors of the (dimensionless) position operator $\hat{x} = (\hat{a}^* + \hat{a})/\sqrt{2}$, with continuum spectrum \mathbb{R} (see for instance [25]). Projection on this set provides a space of complex valued functions $\psi(x) = \langle x|\psi\rangle$. The finite norm of $|\psi\rangle$ in \mathcal{F} implies that $\psi(x)$ is square integrable on \mathbb{R} with the Lebesgue measure dx . The inner product here is realized by

$$(\psi', \psi) := \int_{\mathbb{R}} \overline{\psi'(x)} \psi(x) dx = \langle \psi' | \psi \rangle.$$

Then this realization leads to the linear space $L^2(\mathbb{R}, dx)$. This is the most usual representation in physics, known as coordinate representation, while its elements are called wave functions. Notice that $|x\rangle$ is not a vector in \mathcal{F} , but in the corresponding Rigged Hilbert space.

$F^2(\mathbb{C}, d\mu)$: Segal-Bargmann space: consider the set

$$\{|z\rangle : \hat{a}|z\rangle = z|z\rangle\} \tag{3}$$

containing the normalized eigenvectors of \hat{a} , with continuum spectrum \mathbb{C} (see for instance [24]). The vectors $|z\rangle$ are called coherent states (or Poisson vectors) and form an overcomplete set in \mathcal{F} (see the Appendix for technical details). Projection on this set provides complex valued functions $\langle \bar{z}|\psi\rangle$. It is convenient to write

$$\psi(z) = \langle \bar{z}|\psi\rangle e^{\frac{1}{2}|z|^2}, \tag{4}$$

called Segal-Bargmann symbol for $|\psi\rangle \in \mathcal{F}$ in the following, because the finite norm of $|\psi\rangle$ in \mathcal{F} implies $\psi(z)$ is an entire function. Moreover, $\psi(z)$ is square integrable with the Gaussian

measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} d^2z$ (d^2z being the Lebesgue translationally invariant measure on \mathbb{C}). This realization, called the Segal-Bargmann space $F^2(\mathbb{C}, d\mu)$, is then the subset of *entire* functions in $L^2(\mathbb{C}, d\mu)$, the Hilbert space of square integrable functions on \mathbb{C} with the Gaussian measure $d\mu(z)$. The inner product in $F^2(\mathbb{C}, d\mu)$ is realized by

$$(\psi', \psi) := \int_{\mathbb{C}} \overline{\psi'(z)} \psi(z) d\mu(z) = \langle \psi' | \psi \rangle.$$

The Segal-Bargmann functional realization $F^2(\mathbb{C}, d\mu)$, where Toeplitz operators are defined, has as important advantages the connection with the powerful theory of analytic functions, and the fact that vectors $|z\rangle \in \mathcal{F}$ (in contrast with $|x\rangle \notin \mathcal{F}$). However, the set of coherent states is not an orthogonal basis but an overcomplete set in \mathcal{F} . Completeness means that vectors in \mathcal{F} can be written as

$$\int_{\mathbb{C}} d\mu(z) f(z, \bar{z}) |z\rangle. \quad (5)$$

Indeed, one can show that such a linear combination is a vector in \mathcal{F} if and only if $f(z, \bar{z}) \in L^2(\mathbb{C}, d\mu)$. Overcompleteness means that the function $f(z, \bar{z})$ is not unique for a given vector $|\psi\rangle \in \mathcal{F}$; all the functions in $L^2(\mathbb{C}, d\mu)$ expanding the same vector define equivalence classes, but there exists a unique entire function $\psi(z) \in F^2(\mathbb{C}, d\mu)$ representing each class, exhibiting the isomorphism between \mathcal{F} and $F^2(\mathbb{C}, d\mu)$. Given a vector $|\psi\rangle$, such a function is just the Segal-Bargmann symbol $\psi(z)$ of the vector, defined in eq. (4). These properties are easily derived using the well known Bargmann projection, defined in $L^2(\mathbb{C}, d\mu)$ as follows: given $g \in L^2(\mathbb{C}, d\mu)$, let

$$K_{\bar{z}}(w) := e^{\bar{z}w} \quad (6)$$

and

$$(P g)(z) = (K_{\bar{z}}, g), \quad (7)$$

with the inner product in $L^2(\mathbb{C}, d\mu)$. It holds that

$$P : L^2(\mathbb{C}, d\mu) \rightarrow F^2(\mathbb{C}, d\mu);$$

this feature is essential to the definition of Toeplitz operators given below in eq. (12).

From the description above, there exist natural unitary isomorphisms between l^2 , $L^2(\mathbb{R}, dx)$ and $F^2(\mathbb{C}, d\mu)$. Indeed, since the Fock space is a unitary irreducible representation of the Heisenberg-Weyl group [24], these isomorphisms are just examples of the celebrated theorem by Stone and

von Neumann [27, 28] stating that any two unitary irreducible representations of the Heisenberg-Weyl group are unitarily equivalent. It is then convenient to use the different realizations for the purposes to which they are most suited. All the results proved for one realization, if formulated only in terms of a vacuum vector and annihilation and creation operators, are then valid for the others.

Let us recall that the natural isomorphism between $F^2(\mathbb{C}, d\mu)$ and l^2 , that will be used throughout the present work, is derived from the relation

$$\langle \bar{z}|n\rangle = e^{-\frac{1}{2}|z|^2} \frac{z^n}{\sqrt{n!}} \quad (8)$$

(see the Appendix) between vectors in the canonical basis (1) and the coherent states set (3). A vector $|\psi\rangle$, as given in (2), is realized in l^2 by $\{\psi_n\}$ and in $F^2(\mathbb{C}, d\mu)$ by $\psi(z) = \sum_{n \in \mathbb{N}} \psi_n \frac{z^n}{\sqrt{n!}}$.

We summarize this relations in the following:

Proposition II.1 *Let $U : F^2(\mathbb{C}, d\mu) \rightarrow l^2$ be defined by*

$$U\psi = \{\psi^{(n)}(0)/\sqrt{n!}\}. \quad (9)$$

Then U is a unitary isomorphism between $F^2(\mathbb{C}, d\mu)$ and l^2 .

Proof: Considering the Maclaurin expansion of $\psi \in F^2(\mathbb{C}, d\mu)$, it is straightforward to compute

$$(\psi, \psi) = \sum_{n \in \mathbb{N}} \frac{|\psi^{(n)}(0)|^2}{n!},$$

showing that $\{\psi_n\} \equiv \{\psi^{(n)}(0)/\sqrt{n!}\} \in l^2$. Because of uniqueness of the expansion coefficients, $U^{-1} : l^2 \rightarrow F^2(\mathbb{C}, d\mu)$ exists and is given by

$$(U^{-1}\{\psi_n\})(z) = \sum_{n \in \mathbb{N}} \frac{\psi_n}{\sqrt{n!}} z^n,$$

with the series converging in \mathbb{C} .

Finally, it is easy to compute for any $\psi', \psi \in F^2(\mathbb{C}, d\mu)$ that

$$(U\psi', U\psi) = \sum_{n \in \mathbb{N}} \overline{\psi'_n} \psi_n = (\{\psi'_n\}, \{\psi_n\})$$

showing that U is unitary.

III. ANTI-WICK AND TOEPLITZ OPERATORS

We now introduce a class of integral operators on \mathcal{F} formally given by a “diagonal” expression in the coherent states set. Namely, given a measurable function $\varphi(w, \bar{w})$, not necessarily analytic, let

$$A_\varphi = \int_{\mathbb{C}} \frac{d^2 w}{\pi} |\bar{w}\rangle \varphi(w, \bar{w}) \langle \bar{w}| \quad (10)$$

where the notation $\langle \bar{w}|$ stands for the linear form $\langle \bar{w}|\cdot\rangle$ on \mathcal{F} and the integral is understood in the weak sense. The function $\varphi(w, \bar{w})$ is known as the anti-Wick or contravariant symbol of A_φ [4], and we refer to A_φ as an anti-Wick operator. A simple example is the (dimensionless) Hamiltonian operator for a harmonic oscillator, $\hat{H} = \hat{a}^* \hat{a} + 1/2$, whose anti-Wick symbol results $H(w, \bar{w}) = w\bar{w} - 1/2$; such an operator form can indeed be obtained for any polynomial operator in \hat{a} and \hat{a}^* [6].

For a given symbol, the formal expression (10) defines an operator $A_\varphi : D \subseteq \mathcal{F} \rightarrow \mathcal{F}$ with non trivial domain D when the symbol is suitably restricted. For instance, if, for some $r > 2$, $\int_{\mathbb{C}} |\varphi(w, \bar{w})|^r \exp(-|w|^2) \frac{d^2 w}{\pi} < \infty$, then A_φ is well defined on a dense domain in \mathcal{F} [24].

Being written in terms of coherent state vectors, the most natural realization of A_φ is that on $F^2(\mathbb{C}, d\mu)$. Projecting $A_\varphi|\psi\rangle$ on $|\bar{z}\rangle$,

$$\langle \bar{z}|A_\varphi|\psi\rangle = \int_{\mathbb{C}} \frac{d^2 w}{\pi} \langle \bar{z}|\bar{w}\rangle \varphi(w, \bar{w}) \langle \bar{w}|\psi\rangle \quad (11)$$

leads (see the Appendix) to the following expression for its Segal-Bargmann symbol:

$$\int_{\mathbb{C}} d\mu(w) \overline{K_{\bar{z}}(w)} \varphi(w, \bar{w}) \psi(w). \quad (12)$$

Let us call

$$T_\varphi : D \subseteq F^2(\mathbb{C}, d\mu) \rightarrow F^2(\mathbb{C}, d\mu)$$

the realization of A_φ acting on $F^2(\mathbb{C}, d\mu)$. As the integral in (12) is the inner product in $L^2(\mathbb{C}, d\mu)$ defining the Bargmann projection (7), then

$$(T_\varphi \psi)(z) = (P \varphi \psi)(z). \quad (13)$$

This last equation *defines* a Toeplitz operator on $F^2(\mathbb{C}, d\mu)$ (see for instance [9]).

According to original works by Berezin [4], for a given symbol φ the *natural domain* of T_φ is the subset of $F^2(\mathbb{C}, d\mu)$

$$Dom(T_\varphi) = \{\psi \in F^2(\mathbb{C}, d\mu) : \varphi \psi \in L^2(\mathbb{C}, d\mu)\}. \quad (14)$$

Then the Bargmann projection is well defined, warranting that the integral in eq. (12) is an inner product in $L^2(\mathbb{C}, d\mu)$, with result in $F^2(\mathbb{C}, d\mu)$. Following this line, one can analyze classes of symbols such that the natural domain is wide enough in $F^2(\mathbb{C}, d\mu)$. For instance Berger and Coburn [7] developed a symbolic calculus for Toeplitz operators with bounded symbols, whose natural domains are the whole space $F^2(\mathbb{C}, d\mu)$. Later, Coburn [13] defined a class of symbols φ such that

$$\forall z \in \mathbb{C} \quad \varphi(w, \overline{w})e^{w\overline{z}} \in L^2(\mathbb{C}, d\mu); \quad (15)$$

in other words, the Segal-Bargman symbols for all the coherent states belong to the natural domain, making it dense in $F^2(\mathbb{C}, d\mu)$. In the same spirit, Folland [16] considers safe to work with a class of symbols φ such that

$$|\varphi(w, \overline{w})| \leq C \exp(\delta|w|^2), \text{ with } \delta < 1/2, \quad (16)$$

which are included in Coburn's class.

However, the standard domain definition (14) is somewhat restrictive, as eq. (12) may make sense as a well defined integral even when it is not an inner product in $L^2(\mathbb{C}; d\mu)$. Some authors, as Janas [21], consider the largest domain in $F^2(\mathbb{C}, d\mu)$:

$$\{\psi \in F^2(\mathbb{C}, d\mu) : \int_{\mathbb{C}} d\mu(w) \overline{K(w, \overline{z})} \varphi(w, \overline{w}) \psi(w) \in F^2(\mathbb{C}, d\mu)\}, \quad (17)$$

which in some cases may be indeed larger than $\text{Dom}(T_\varphi)$ [22].

In what follows, we adhere to the natural domain definition (14) given by Berezin.

Besides the realization in the Segal-Bargmann space $F^2(\mathbb{C}, d\mu)$, for a given symbol $\varphi(w, \overline{w})$ one could also consider the realization of the anti-Wick operator (10) on the other isometric spaces. It is well known that the realization in $L^2(\mathbb{R}, dx)$ leads to pseudo-differential operators in Weyl form [19], a setting where powerful tools are available. The realization on l^2 has also been considered, in particular in [18], and turns out to be most convenient for Toeplitz operators with radial symbols. It presents some features that will be analyzed in Sections V and VI.

IV. RADIAL ANTI-WICK AND TOEPLITZ OPERATORS

We consider in this Section anti-Wick or Toeplitz operators with radial symbols, namely $\varphi(w, \overline{w}) = \varphi(|w|)$. A salient feature of such operators is that any vector in the canonical basis (1), also belonging to the operator domain, is an eigenvector. In order to discuss this property and its consequences, we find it convenient to give the following

Definition IV.1 We denote \mathcal{P} the class of radial symbols

$$\mathcal{P} = \{\varphi(|w|) : \forall n \in \mathbb{N}, u_n \in \text{Dom}(T_\varphi)\} \quad (18)$$

where $u_n(z) = z^n / \sqrt{n!}$ are the Segal-Bargmann symbols for vectors $|n\rangle$ in the orthonormal complete set (1), and $\text{Dom}(T_\varphi)$ is the natural domain defined in (14).

For symbols $\varphi(|w|)$ in this class, the Toeplitz operator T_φ has a dense natural domain, including at least any polynomial in $F^2(\mathbb{C}, d\mu)$, as well as A_φ is well defined on any finite linear combination of vectors $|n\rangle$.

In connection with the classes (15, 16) described in Section III, we mention the following

Proposition IV.2 If $\varphi(|w|)$ is a radial symbol in the Coburn's class (15), then $\varphi(|w|) \in \mathcal{P}$

Proof: This relation is proven in Theorem 1.3 (i) in [22], even for non-radial symbols. It is also proven that the converse is not true.

Let $\varphi(|w|) \in \mathcal{P}$ and consider the radial anti-Wick operator

$$A_\varphi = \int_{\mathbb{C}} \frac{d^2 w}{\pi} |\bar{w}\rangle \varphi(|w|) \langle \bar{w}| \quad (19)$$

Then the matrix elements $\langle m|A_\varphi|n\rangle$ in the canonical basis (1) can be computed. The key feature of radial symbols is that the integrals over the complex plane are easily solved in polar coordinates: using eq. (8) it is straightforward to show that

$$\langle m|A_\varphi|n\rangle = \int_{\mathbb{C}} d\mu(w) \varphi(|w|) \frac{\bar{w}^m}{\sqrt{m!}} \frac{w^n}{\sqrt{n!}} = \delta_{mn} \varphi_n, \quad (20)$$

where

$$\varphi_n = \frac{2}{n!} \int_0^\infty \varphi(r) r^{2n+1} e^{-r^2} dr. \quad (21)$$

As the off-diagonal elements vanish the anti-Wick operator is drastically simplified, with a diagonal expression

$$A_\varphi = \sum_{n \in \mathbb{N}} |n\rangle \varphi_n \langle n| \quad (22)$$

in the canonical Fock space basis (1). This of course provides its spectral decomposition, with eigenvalues φ_n and eigenvectors $|n\rangle$ in (1).

From this decomposition, it is most suited to consider the realization of A_φ on l^2 : given a Fock space vector $|\psi\rangle = \sum_{n \in \mathbb{N}} \psi_n |n\rangle$, one gets

$$A_\varphi |\psi\rangle = \sum_{n \in \mathbb{N}} \varphi_n \psi_n |n\rangle. \quad (23)$$

While $|\psi\rangle$ is realized by the sequence $\{\psi_n\} \in l^2$, its image $T_\varphi |\psi\rangle$ is realized by the sequence $\{\varphi_n \psi_n\}$. Introducing the notation D_φ for the realization of A_φ on l^2 , we then have

$$D_\varphi \{\psi_n\} = \{\varphi_n \psi_n\}, \quad (24)$$

that is D_φ acts as a pointwise multiplication operator on l^2 sequences.

Consider now the realization of the same anti-Wick operator, with symbol $\varphi(|w|) \in \mathcal{P}$ as a radial Toeplitz operator T_φ on $F^2(\mathbb{C}, d\mu)$. As a consequence of the isomorphism U in Proposition II.1, it is apparent that the Toeplitz operator T_φ on $F^2(\mathbb{C}, d\mu)$ is unitarily equivalent to the diagonal operator D_φ on l^2 .

The l^2 realization of radial Toeplitz operators has been used in [3, 13] and elsewhere to analyze simple examples, while it has been exploited by Grudsky and Vasilevski in [18] as the key point to analyze boundedness, compactness and spectral properties of radial Toeplitz operators. However, the equivalence between Toeplitz operators with radial symbols and diagonal operators of the form (24) must be treated carefully; in particular, an unattentive reading of [18] can be misleading, as operator domains are not explicitly regarded. To be precise, let us denote by $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, as in [18], the set of all measurable functions $\phi(r)$ on \mathbb{R}_+ such that

$$\int_0^\infty |\phi(r)| r^m e^{-r^2} dr < \infty. \quad (25)$$

There exist symbols $\phi(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ but $\phi(|w|) \notin \mathcal{P}$ (see Example IV.5 below). Though the matrix elements in (20) can not be computed for such symbols, one still can compute a sequence $\{\phi_n\}$ as in (21). We then give the following

Definition IV.3 *Given a radial symbol $\phi(|w|)$ in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, let $\tilde{D}_\phi : D \subset l^2 \rightarrow l^2$ be defined by*

$$\tilde{D}_\phi \{\psi_n\} := \{\phi_n \psi_n\}, \quad (26)$$

with

$$\phi_n := \frac{2}{n!} \int_0^\infty \phi(r) r^{2n+1} e^{-r^2} dr. \quad (27)$$

Notice that we define the domain D of \tilde{D}_ϕ with the condition $\{\phi_n\psi_n\} \in l^2$. D is dense in l^2 , as \tilde{D}_ϕ is well defined at least on finite sequences, but for some infinite sequence $\{\phi_n\psi_n\}$ could lie outside l^2 .

Now, if a radial symbol $\phi(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ but $\phi(|w|) \notin \mathcal{P}$, the operator \tilde{D}_ϕ has a dense domain in l^2 and then can not be equivalent to T_ϕ . This remark should be contrasted with Theorem 3.1 in [18]. In order to complete the analysis given there, we summarize the above discussion in the following:

Theorem IV.4 *Given a radial symbol $\varphi(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, the Toeplitz operator T_φ defined in (13) is unitarily equivalent to the diagonal operator \tilde{D}_φ defined in (IV.3) if and only if $\varphi(|w|) \in \mathcal{P}$. In the positive case, the equivalence is given by $T_\varphi = U^{-1}\tilde{D}_\varphi U$, with the unitary operator U defined in (9).*

Proof: If $\varphi(|w|) \in \mathcal{P}$, it is enough to consider the action of $U^{-1}\tilde{D}_\varphi U$ on functions u_m as given in the definition IV.1.

$$(U^{-1}\tilde{D}_\varphi U u_m)(z) = (U^{-1}\tilde{D}_\varphi)(\{\delta_{mn}\}_{n \in \mathbb{N}}) = U^{-1}(\{\varphi_m \delta_{mn}\}_{n \in \mathbb{N}}) = \varphi_m u_m(z)$$

On the other hand, it is straightforward to compute $(T_\varphi u_m)(z)$ using polar coordinates, getting the same result

$$(T_\varphi u_m)(z) = \varphi_m u_m(z).$$

Turning to the case where $\varphi(|w|) \notin \mathcal{P}$, there exists $u_m \notin \text{Dom}(T_\varphi)$. But $U^{-1}\tilde{D}_\varphi U$ is well defined on u_m , then T_φ and $U^{-1}\tilde{D}_\varphi U$ have different domains and in consequence are not equal.

□

It is important to show that the negative situation in Theorem IV.4 is indeed possible with the following:

Example IV.5 *Consider the function $\varphi(r) = e^{(\frac{1}{2} + \frac{\sqrt{3}}{2}i)r^2}$. It belongs to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, with an associated sequence $\varphi_n = (\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-(n+1)}$. Moreover, this sequence is bounded ($|\varphi_n| = 1$) and the operator (26) is bounded, with domain l^2 . However, $\varphi(|w|) \notin \mathcal{P}$: it is enough to observe that $u_0(z) = 1$ does not belong to $\text{Dom}(T_\varphi)$.*

We stress that, if $\varphi(|w|) \in \mathcal{P}$, then \tilde{D}_φ is nothing but the realization on l^2 of the formal anti-Wick operator A_φ realized by the Toeplitz operator T_φ on $F^2(\mathbb{C}, d\mu)$. On the other hand, if $\varphi(|w|) \notin \mathcal{P}$, one can consider the following

Definition IV.6 Given a radial symbol $\varphi(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, let

$\tilde{T}_\varphi : D \subseteq F^2(\mathbb{C}, d\mu) \rightarrow F^2(\mathbb{C}, d\mu)$ be defined by

$$\tilde{T}_\varphi := U^{-1} \tilde{D}_\varphi U \quad (28)$$

with \tilde{D}_φ given in IV.3 and U defined in (9), with domain

$$D = \text{Dom}(\tilde{T}_\varphi) = \left\{ f \in F^2(\mathbb{C}, d\mu) : \tilde{D}_\varphi U f \in l^2 \right\}.$$

In order to motivate this definition we prove the following:

Theorem IV.7 Let T_φ be a radial Toeplitz operator with symbol φ in

$L_1^\infty(\mathbb{R}_+, e^{-r^2})$, and let \tilde{T}_φ be defined in IV.6. Then

a - $\text{Dom}(T_\varphi) \subseteq \text{Dom}(\tilde{T}_\varphi)$, and

b - the restriction of \tilde{T}_φ to $\text{Dom}(T_\varphi)$ coincides with T_φ .

Proof:

Let $\varphi \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, and let $f(z) \in \text{Dom}(T_\varphi)$. As $f(z) \in F^2(\mathbb{C}, d\mu)$, it can be expanded as $f(z) = \sum_{n \in \mathbb{N}} a_n u_n(z)$ with $\{a_n\} \in l^2$. Let $g(z) = (T_\varphi f)(z)$, also in $F^2(\mathbb{C}, d\mu)$, be expanded as $g(z) = \sum_{n \in \mathbb{N}} b_n u_n(z)$ with $\{b_n\} \in l^2$. The coefficients b_n can be computed with the inner product in $F^2(\mathbb{C}, d\mu)$ as $b_n = (u_n, g)$; using polar coordinates one easily obtains

$$b_n = (u_n, g) = \varphi_n a_n, \quad (29)$$

showing that $\{\varphi_n a_n\} \in l^2$.

a - To see that $f \in \text{Dom}(\tilde{T})$, compute

$$\tilde{T}_\varphi f = U^{-1} \tilde{D}_\varphi U f = U^{-1} \tilde{D}_\varphi \{a_n\} = U^{-1} \{\varphi_n a_n\}.$$

As $\{\varphi_n a_n\} \in l^2$, then $f \in \text{Dom}(\tilde{T}_\varphi)$. Moreover,

$$\tilde{T}_\varphi f = \sum_{n \in \mathbb{N}} \varphi_n a_n u_n(z) \in F^2(\mathbb{C}, d\mu).$$

b - For $f(z) \in \text{Dom}(T_\varphi)$ we have computed above $(T_\varphi f)(z)$, coinciding with the last equation.

□

From the discussion above, given a radial symbol $\varphi(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ but $\varphi(|w|) \notin \mathcal{P}$, one can propose \tilde{T}_φ to be considered a nontrivial extension of the Toeplitz operator T_φ , as in this case the inclusion $\text{Dom}(T_\varphi) \subset \text{Dom}(\tilde{T}_\varphi)$ is strict. To close this Section, we would like to point out that most of the results in [18] apply to such an extension.

V. DIAGONAL OPERATORS ON l^2

As a radial Toeplitz operator can, under certain assumptions, be faithfully mapped into a diagonal operator on l^2 , the natural question to consider is the inverse situation. Namely, given a sequence $\{\gamma_n\}$ defining the spectrum of a diagonal operator on the canonical basis of l^2 , we would like to know whether there exists a radial symbol $\gamma(|w|)$ defining an equivalent Toeplitz operator. But in the Segal-Bargmann space the answer to this question should as a first step invert eq. (27) to construct the symbol $\gamma(|w|)$, and, as a second step, establish whether this symbol provides a Toeplitz operator defined on any polynomial or not, as discussed in the previous Section.

The first step was partially solved in [18]: by means of a nice construction based on analytic continuation and Fourier transforms, the authors show that for any *bounded* sequence $\{\gamma_n\}$ there exists a radial symbol $\gamma(|w|)$ in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that the sequence defined in (27) is again $\{\gamma_n\}$. However, a symbol in this set does not guarantee the equivalence of a Toeplitz operator T_γ with a diagonal operator \tilde{D}_γ . In an attempt to complete the second step we found that, unfortunately, the symbol provided in [18] is hard to analyze in full generality. Moreover, as far as we know there are no results for *unbounded* sequences.

In this section we aim to provide some insight on these questions, by restricting to a family of sequences $\{k^{-n}\}_{n \in \mathbb{N}}$ for which a symbol γ_k in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ can be recognized with a close expression:

Proposition V.1 *Let $k \in \mathbb{C}$. If $\operatorname{Re}(k) > 0$, the sequence $\{k^{-n}\}$ is obtained by (27) from the radial symbol $\gamma_k(|w|) = ke^{(1-k)|w|^2} \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$.*

Proof: The computation is straightforward.

It is simple to explore some properties of sequences in V.1:

Proposition V.2 *The symbol $\gamma_k(|w|)$ is in the class \mathcal{P} if and only if $\operatorname{Re}(k) > 1/2$. Correspondingly, the Toeplitz operator T_{γ_k} is unitarily equivalent to \tilde{D}_{γ_k} defined in IV.3 if and only if $\operatorname{Re}(k) > 1/2$.*

Proof: according to the definition in (18), it is easy to check that for all $n \in \mathbb{N}$

$$\int d\mu(w) |\gamma_k(|w|) u_n(w)|^2 < \infty$$

if and only if $\operatorname{Re}(k) > 1/2$. From Theorem IV.4, T_{γ_k} is then unitarily equivalent to \tilde{D}_{γ_k} .

Relatedly, the same condition on k characterizes whether the symbol γ_k is in the classes considered by Folland and Coburn:

Proposition V.3

a - The symbol γ_k is in Folland's class (16) if and only if $\operatorname{Re}(k) > 1/2$.

b - The symbol γ_k is in Coburn's class (15) if and only if $\operatorname{Re}(k) > 1/2$.

Proof:

a - notice that $\gamma_k(|w|) = ke^{i\operatorname{Im}(k)|w|^2}e^{\delta|w|^2}$ with $\delta = 1 - \operatorname{Re}(k)$.

b - as in the previous proposition, it is easy to check that for all $z \in \mathbb{C}$

$$\int d\mu(w) |\gamma_k(|w|)K_z(w)|^2 < \infty$$

if and only if $\operatorname{Re}(k) > 1/2$.

As a consequence, our symbols $\gamma_k(|w|)$ belong simultaneously to the classes \mathcal{P} , Folland's and Coburn's, or to none of them. While no subtle differences between these classes will be found in our analysis, the results in this Section apply to all of them. Notice that a well known operator in Quantum Mechanics, the Maxwell-Boltzman density matrix for a harmonic oscillator, can be written as a Toeplitz operator with this kind of symbol. The operator is defined as

$$\hat{\rho} = e^{-\beta(\hat{a}^*\hat{a}+1/2)},$$

where $\beta > 0$ is the (dimensionless) inverse temperature. As

$$\hat{\rho}|n\rangle = e^{-\beta/2}e^{-\beta n}|n\rangle,$$

the operator is diagonal on l^2 with a power like sequence spectrum

$\{e^{-\beta/2}(e^\beta)^{-n}\}$. Being $e^\beta > 1$, the corresponding anti-Wick symbol reads $\rho(|w|) = e^{\beta/2}e^{(1-e^\beta)|w|^2}$ and belongs to \mathcal{P} .

We explore first bounded sequences $\{k^{-n}\}$, with $|k| \geq 1$.

Proposition V.4 *If $\operatorname{Re}(k) > 1/2$ and $|k| \geq 1$, the symbol $\gamma_k(|w|)$ coincides with the one constructed by Grudsky and Vasilevski in the Theorem 3.7 of [18]. Moreover, the Toeplitz operator T_{γ_k} is then unitarily equivalent to \tilde{D}_{γ_k} .*

Proof: as $|k| \geq 1$, the sequence $\{k^{-n}\}$ is bounded and Theorem 3.7 of [18] provides a radial symbol in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$. As $\operatorname{Re}(k) > 1/2$, uniqueness of the symbol for a given Toeplitz operator follows as a known property of Folland's class [16]. Also equivalence between T_{γ_k} and \tilde{D}_{γ_k} follows from Theorem IV.4.

Proposition V.5 *If $|k| > 1$ and $0 < \operatorname{Re}(k) < 1/2$, the symbol γ_k belongs to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ but the natural domain $\operatorname{Dom}(T_{\gamma_k})$ is trivial.*

Proof: As mentioned in Proposition V.1, $\operatorname{Re}(k) > 0$ implies that $\gamma_k(|w|) \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. For $f \in F^2(\mathbb{C}, d\mu)$ being in the natural domain of T_{γ_k} , the condition $\gamma_k f \in L^2(\mathbb{C}, d\mu)$ requires $|f(z)|$ to decay exponentially; then, by Liouville theorem, the only function in $\operatorname{Dom}(T_{\gamma_k})$ is $f(z) = 0$.

Turning to unbounded sequences, $|k| < 1$, there is no general statement about the existence of an equivalent Toeplitz operator. However, for the family of symbols under consideration, we learn that:

Proposition V.6 *If $|k| < 1$ and $\operatorname{Re}(k) > 1/2$, the sequence $\{k^{-n}\}$ is unbounded but the symbol $\gamma_k \in \mathcal{P}$ still allows the definition of an associated \tilde{D}_{γ_k} as defined in IV.3. Then the Toeplitz operator T_{γ_k} is densely defined, unbounded and unitarily equivalent to the diagonal operator \tilde{D}_{γ_k} on l^2 .*

VI. COMPOSITION OF TOEPLITZ OPERATORS: AN OPEN PROBLEM.

A problem still open is to determine when the composition of Toeplitz operators is closed. That is, given symbols φ and η in certain class, to determine the existence of a symbol τ such T_τ is a well defined Toeplitz operator and $T_\tau = T_\varphi T_\eta$.

A first approach to this question was given by Coburn in [12], where symbols in a smooth Bochner algebra are considered. These are Fourier-Stieltjes transforms of compactly supported, regular, bounded complex-valued Borel measures on \mathbb{C} , indeed a subclass of the symbols defined in (15). The author proved that this class is closed under composition: $T_\varphi T_\eta$ is a Toeplitz operator with a symbol τ also in the smooth Bochner algebra. The composition symbol τ can be calculated as a Moyal type product $\tau = \varphi \diamond \eta$ given by

$$(\varphi \diamond \eta)(w, \bar{w}) = \sum_k \frac{(-1)^k}{k!} \partial^k \varphi(w, \bar{w}) \bar{\partial}^k \eta(w, \bar{w}). \quad (30)$$

where $\partial^k := \frac{\partial^k}{\partial^k w}$ and $\bar{\partial}^k := \frac{\partial^k}{\partial^k \bar{w}}$. Later, in [13] this result was extended to unbounded Toeplitz operators with polynomial symbols in w and \bar{w} .

More recently Bauer [2] considered in detail the class of symbols in the set

$$\operatorname{Sym}_{>0}(\mathbb{C}) := \bigcap_{j=1}^{\infty} \mathcal{D}_{1/j} \quad (31)$$

where $\mathcal{D}_c := \{\varphi : \exists d > 0 \text{ such that } |\varphi(z, \bar{z})| \leq d \exp(c|z|^2) \text{ a.e.}\}$, providing well defined unbounded Toeplitz operators with domain on a certain scale of Banach spaces. The author proved that the composition of Toeplitz operators is closed for this class, with a symbol for the composition given by the same product (30). Notice that polynomial symbols are included here providing an independent proof of Coburn's results.

For Toeplitz operators with radial symbols, the composition problem can be simplified using the results in Section IV. Given two radial symbols φ, η in the class \mathcal{P} , from Theorem IV.4 one has

$$T_\varphi = U^{-1} \tilde{D}_\varphi U \quad T_\eta = U^{-1} \tilde{D}_\eta U \quad (32)$$

so that

$$T_\varphi T_\eta = U^{-1} \tilde{D}_\varphi \tilde{D}_\eta U, \quad (33)$$

where the composition of diagonal operators on l^2 is trivially expressed as

$$\tilde{D}_\varphi \tilde{D}_\eta \{c_n\} = \{\varphi_n \eta_n c_n\}. \quad (34)$$

The domain of this operator is clearly dense in l^2 , so that the composition $T_\varphi T_\eta$ in (33) is well defined on a dense domain in $F^2(\mathbb{C}, d\mu)$. It is also clear that the composition is commutative in this case. However, $U^{-1} \tilde{D}_\varphi \tilde{D}_\eta U$ does not necessarily correspond to a radial Toeplitz operator with some symbol τ . From the results in Section IV, can summarize the following conditions:

Proposition VI.1 *Let φ, η be radial symbols in the class \mathcal{P} . If there exists a radial symbol τ in \mathcal{P} such that the sequence $\{\tau_n\}$ given by (27) coincides with $\{\varphi_n \eta_n\}$, then $T_\tau = T_\varphi T_\eta$, that is the composition of radial Toeplitz operators is another radial Toeplitz operator. If not, the composition of such radial Toeplitz operators is not a Toeplitz operator with symbol in \mathcal{P} .*

There are known counter examples [2]-[13] that exhibit such limitations on the possibility of composing Toeplitz operators with symbols in some given class. In particular, [13] discusses an example with a radial symbol which is indeed in the family that we considered in Section V (with $k = \frac{3}{5} - \frac{4}{5}i$). We can give some generality to this analysis, within the symbols in that family, with the following:

Proposition VI.2 *Let $\gamma_a(|w|) = ae^{(1-a)|w|^2}$ and $\gamma_b(|w|) = be^{(1-b)|w|^2}$ with $Re(a) > 1/2$ and $Re(b) > 1/2$. The operator $T_{\gamma_a} T_{\gamma_b}$ is a Toeplitz operator with radial symbol in \mathcal{P} if and only if $Re(ab) > 1/2$. In the positive case, the symbol is $\gamma_c = ce^{(1-c)|w|^2}$ with $c = ab$.*

Proof: From V.1 and V.2, T_φ and T_η are unitarily equivalent to \tilde{D}_φ and \tilde{D}_η , described by sequences $\{a^{-n}\}$ and $\{b^{-n}\}$, respectively. Their composition is described by the sequence $\{(ab)^{-n}\}$. The condition on $Re(ab)$ and the symbol γ_c are finally read from V.1 and V.2.

It is interesting to note that:

Proposition VI.3 *The Moyal product (30) given by Coburn for polynomial and Bochner algebra symbols provides the correct symbol for the composition described in Proposition VI.2, with $Re(a.b) > 1/2$.*

Proof: The computation of the Moyal product is straightforward, and the resulting series can be summed to give the expected result. Indeed, the product can be done for any $a, b \in \mathbb{C}$ and is commutative, but when the symbols are not in \mathcal{P} one can not associate densely defined Toeplitz operators to them.

The composition of Toeplitz operators is then not closed for the family of symbols $ke^{(1-k)|w|^2} \in \mathcal{P}$ with $Re(k) > 1/2$. However, particular examples where composition of Toeplitz operators not belonging to closed classes is a Toeplitz operator can be given. For instance, pick $a = \frac{3}{5} - \frac{4}{5}i$ and $b = \frac{3}{5} + \frac{4}{5}i$, so that $ab = 1$, $T_{\gamma_a}T_{\gamma_b} = T_{\gamma_1}$ is the identity in $F^2(\mathbb{C}, d\mu)$ and all the involved symbols are in \mathcal{P} .

One is tempted now to look for a class of radial symbols where the composition of the corresponding Toeplitz operators is closed. Within the scope of Proposition V.1, we are forced to restrict to the following:

Definition VI.4 *Let*

$$\mathcal{L} = \left\{ \gamma_k(|w|) = ke^{(1-k)|w|^2} : k \in \mathbb{R} \wedge k \geq 1 \right\}.$$

From Proposition VI.2, it is immediate that:

Proposition VI.5 *The class of radial symbols \mathcal{L} is closed under composition of the corresponding Toeplitz operators.*

Unfortunately, this result is not new, as symbols in \mathcal{L} are bounded. They are in Bauer's class defined in (31), and also in the (not smooth) Bochner algebra, being the Fourier-Stieltjes of Gaussian measures with non compact support.

We conclude this section reviewing the mentioned example in [13], the Toeplitz operator T_{γ_a} with $a = \frac{3}{5} - \frac{4}{5}i$. The symbol $\gamma_a(|w|)$ is not bounded, but the natural domain of T_{γ_a} contains all of the coherent states, that is $\gamma_a(|w|)$ belongs to the class defined in (15). Coburn proves that the composition $T_{\gamma_a}T_{\gamma_a}$ can not be written as a Toeplitz operator with a symbol in the same class. In our notation, as $\operatorname{Re}(a) > 1/2$, we can write

$$T_{\gamma_a} = U\tilde{D}_{\gamma_a}U^{-1},$$

where $\tilde{D}_{\gamma_a}\{\psi_n\} = \{a^{-n}\psi_n\}$. As $|a| = 1$, the spectrum of \tilde{D}_{γ_a} (equivalently T_{γ_a}) is bounded. Then the natural domain of T_{γ_a} is the full space $F^2(\mathbb{C}, d\mu)$. The composition of T_{γ_a} with itself reads

$$T_{\gamma_a}T_{\gamma_a} = U\tilde{D}_{\gamma_a}\tilde{D}_{\gamma_a}U^{-1},$$

well defined on $F^2(\mathbb{C}, d\mu)$ as $\tilde{D}_{\gamma_a}\tilde{D}_{\gamma_a}\{\psi_n\} = \{(a^2)^{-n}\psi_n\}$ has again bounded spectrum. However, $a^2 = \frac{-7}{25} - \frac{24}{25}i$ has negative real part: according to Proposition VI.2, $T_{\gamma_a}T_{\gamma_a}$ is not a Toeplitz operator with symbol in \mathcal{P} , a result that in our framework corresponds to that proved in [13].

Notice that an operator $\Gamma : l^2 \rightarrow l^2$ defined as $\Gamma := \tilde{D}_{\gamma_a}\tilde{D}_{\gamma_a}$ is diagonal, acting as pointwise multiplication by the bounded sequence $\{(a^2)^{-n}\}$. Then, according to Theorem 3.7 in [18], there exists a radial symbol $\varphi(|w|)$ in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that eq. (27) generates the sequence $\{(a^2)^{-n}\}$. This symbol can not belong to \mathcal{P} , so that a Toeplitz operator T_φ with this symbol is not densely defined in $F^2(\mathbb{C}, d\mu)$ and is not equivalent to $T_{\gamma_a}T_{\gamma_a}$.

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APPENDIX: THE STANDARD COHERENT STATES SET

The set of coherent states in eq. (3) is not an orthogonal basis but an overcomplete set in \mathcal{F} [24]. Indeed, one can show that

$$\langle \bar{z}|w\rangle = e^{-\frac{1}{2}|z|^2}e^{-\frac{1}{2}|w|^2}e^{zw}, \quad (35)$$

while the completeness is given by a resolution identity in \mathcal{F} , *i.e.*

$$\mathbb{I} = \int_{\mathbb{C}} \frac{d^2z}{\pi} |z\rangle\langle z| \quad (36)$$

with the integral understood in the weak sense. The notation $\langle z|$ in eq. (36) stands for the linear form $\langle z|\cdot\rangle$ on \mathcal{F} .

Vectors $|z\rangle$ are conveniently related to the vacuum by

$$|z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^*} |0\rangle. \quad (37)$$

Using eq. (37), the operators \hat{a}^* and \hat{a} can be shown to be realized on $F^2(\mathbb{C}, d\mu)$ functions by

$$a^*\psi(z) = z\psi(z), \quad a\psi(z) = \frac{\partial}{\partial z}\psi(z).$$

Then, vectors $|n\rangle$ in the orthonormal basis (1) have Segal-Bargmann symbols

$$u_n(z) = z^n / \sqrt{n!}. \quad (38)$$

The symbol for a normalized coherent state $|w\rangle$ reads, according to eq. (35),

$$u_w(z) = e^{-\frac{1}{2}|w|^2} e^{wz}. \quad (39)$$

It follows from eq. (36) that

$$\psi(z) = \int_{\mathbb{C}} d\mu(w) \overline{K_{\bar{z}}(w)} \psi(w), \quad (40)$$

with $K_{\bar{z}}(w) = e^{\bar{z}w}$ as in eq. (6). The identity in $F^2(\mathbb{C}, d\mu)$ is then represented as an integral operator with kernel $K_{\bar{z}}(w)$, or as an inner product in $F^2(\mathbb{C}, d\mu)$,

$$\psi(z) = (K_{\bar{z}}, \psi). \quad (41)$$

In other words, the restriction of the Bargmann projection (7) to $F^2(\mathbb{C}, d\mu)$ is the identity operator. Thus the existence of the set $\{K_z\}_{z \in \mathbb{C}} \subseteq F^2(\mathbb{C}, d\mu)$ makes it a reproducing kernel Hilbert space. Notice in pass that vectors $K_{\bar{z}}(w)$ are proportional to coherent state vectors $u_{\bar{z}}(w)$ in eq. (39) (some authors prefer to disregard normalization and refer to $K_{\bar{z}}(w)$ as Poisson vectors): eq. (41) can be read as a weighted orthogonal projection of $|\psi\rangle$ on $|\bar{z}\rangle$.

All of the above can be trivially generalized to a finite number N of commuting creation and annihilation pairs, leading to $F^2(\mathbb{C}^N, d\mu)$ spaces. The case of infinite number of coordinates was considered by Segal [26].

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